Finding the Minimal DFA of Very Large Finite State Automata with an Application to Token Passing Networks

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Abstract

Finite state automata (FSA) are ubiquitous in computer science. Two of the most important algorithms for FSA processing are the conversion of a non-deterministic finite automaton (NFA) to a deterministic finite automaton (DFA), and then the production of the unique minimal DFA for the original NFA. We exhibit a parallel disk-based algorithm that uses a cluster of 29 commodity computers to produce an intermediate DFA with almost two billion states and then continues by producing the corresponding unique minimal DFA with less than 800,000 states. The largest previous such computation in the literature was carried out on a 512-processor CM-5 supercomputer in 1996. That computation produced an intermediate DFA with 525,000 states and an unreported number of states for the corresponding minimal DFA. The work is used to provide strong experimental evidence satisfying a conjecture on a series of token passing networks. The conjecture concerns stack sortable permutations for a finite stack and a 3-buffer. The origins of this problem lie in the work on restricted permutations begun by Knuth and Tarjan in the late 1960s. The parallel disk-based computation is also compared with both a single-threaded and multi-threaded RAM-based implementation using a 16-core 128 GB large shared memory computer.

1 Introduction

Finite state automata (FSA) are ubiquitous in mathematics and computer science, and have been studied extensively since the 1950s. Applications include pattern matching, signal processing, natural language processing, speech recognition, token passing networks (including sorting networks), compilers, and digital logic.

This work attempts to relieve the critical bottleneck in many automata-based computations by providing a scalable disk-based parallel algorithm for computing the minimal DFA accepting the same language as a given NFA. This requires the construction of an intermediate non-minimal DFA whose, often very large, size has been the critical limitation on previous RAM-based computations. Thus, researchers may use a departmental cluster or a SAN (storage area network) to produce the desired minimal DFA off-line, and then embed that resulting small DFA in their production application. As a motivating example, Section 6 demonstrates the production of a two-billion state DFA that is then reduced to a minimal DFA with less than 800,000 states — a more than 1,000-fold reduction in size.

As a measure of the power of the technique, we demonstrate an application to the analysis of a series of token passing networks, for which we are now able to complete the experiments needed to conjecture the general properties of the whole series and the infinite "limit" network.

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This disk-based parallel algorithm is based on a RAM-based parallel algorithm used on supercomputers of the 1990s. We adapt that algorithm both to clusters of modern commodity computers and to a multi-threaded algorithm for modern many-core computers. More important, we apply a disk-based parallel computing approach to carry out large computations whose intermediate data would not normally fit within the RAM of commodity clusters. By doing so, we use the subset construction to produce a 2 billion-state intermediate DFA, and then reduce that to a minimal DFA of 3-quarters of a million states. Part of the difficulty of producing the 2 billion-state DFA by the subset construction is that each DFA state consists of a subset that includes up to 20 of the NFA states. Hence, each DFA state needs a representation of 80 bytes (4×20).

The novel contributions of this paper are:

- efficient parallel disk-based versions of known algorithms for determinizing large NFAs (the subset construction) and for minimizing very large DFAs;
- a new multi-threaded implementation for the two algorithms above;
- an application to challenge problems involving stack-sortable permutations encoded as token passing networks; and
- formulation of a conjecture for a series of stack-sortable permutation problems, based on the experimental evidence arising from application to that challenge problem.

This work represents an important advance over the previous state of the art [31], which used a 512-processor CM-5 supercomputer to minimize a DFA with 525,000 states.

In the rest of this paper, Section 2 presents related work. Section 3 presents background on finite state automata and their minimization. It also motivates the importance of the two algorithms (determinization and minimization) by recalling that NFA and DFA form the primary computationally tractable representations for the very important class of regular languages in computer science.

Section 4 then presents the disk-based parallel algorithm for determinization (subset construction) and minimization. It also presents a corresponding multi-threaded computation. Section 5 presents token passing networks and the challenge problem considered here. Section 6 presents the experimental results for the given challenge problem.

2 Related Work

Finite state machines are also an important tool in natural language processing, and have been used for a wide variety of problems in computational linguistics. In a work presenting new applications of finite state automata to natural language processing [26], Mohri cites a number of examples, including: lexical analysis [33]; morphology and phonology [19]; syntax [25, 32]; text-to-speech synthesis [34]; and speech recognition [28, 30]. Speech recognition, in particular, can benefit from the use of very large automata. In [27], Mohri predicted:

> *"More precision in acoustic modeling, finer language models, large lexicon grammars, and a larger vocabulary will lead, in the near future, to networks of much larger sizes in speech recognition. The determinization and minimization algorithms might help to limit the size of these networks while maintaining their time efficiency."*

While the subset construction for determinization has been a standard algorithm since the earliest years, this is not true for the minimization algorithm. For any DFA there is an equivalent minimal canonical DFA [14, Chapter 4.4]. Fast sequential RAM-based DFA minimization algorithms have been developed since the 1950s. A taxonomy of most of these algorithms can be found in [39]. The first DFA minimization algorithms were proposed by Huffman [15] and Moore [29]. Hopcroft's minimization algorithm [13] is proved to achieve the best possible theoretical complexity ($O(|\Sigma|NlogN)$ for alphabet Σ and number of states N).

Hopcroft's algorithm has been extensively revisited [7, 12, 18]. There exist alternative DFA minimization algorithms, such as Brzozowski's algorithm [9], which, for some special cases, performs better in practice than Hopcroft's algorithm [35]. However, none of these sequential algorithms parallelize well (with the possible exception of Brzozowski's, in some cases).

Parallel DFA minimization has been considered since the 1990s. All existing parallel algorithms are for shared memory machines, either using the CRCW PRAM model [37], the CREW pram model [16], or the EREW PRAM model [31]. All of these algorithms are applicable for tightly coupled parallel machines with shared RAM and they make heavy use of random access to shared memory. In addition, [31] minimized a 525,000-state DFA on the CM-5 supercomputer.

When the DFA considered for minimization is very large (possibly obtained from a large NFA by subset construction), it must be stored on disk. To our knowledge, this work represents the first disk-based algorithm for determinization and minimization.

Obtaining a minimal canonical DFA equivalent to a given NFA is important for the analysis of the classes of permutations generated by token passing in graphs. Such a graph is called a *token passing network* (TPN) [3, 5]. This is related to the subject permutations [4], with origins in the 1969 work of Knuth [17, Section 2.2.1] and the 1972 work of Tarjan [36]. TPNs are used to model or approximate a range of data structures, including combinations of stacks, and provide tools for analyzing the classes of permutations that can be sorted or generated using them. Stack sorting problems have been the subject of extensive research [8]. Sorting with two ordered stacks in series is detailed in [6]. Permutation classes defined by TPNs are described in [38]. Very recent work focused on permutations generated by stacks and dequeues [1]. A collection of results on permutation problems expressed as token passing networks is in [24].

3 Terminology and Background

Finite state automata and the closely related concepts of regular languages and regular expressions form a crucial part of the infrastructure of computer science. Among the rich variety of applications of these concepts are natural language grammars, computer language grammars, hidden Markov models, digital logic, transducers, models for object-oriented programming, control systems, and speech recognition.

This section motivates the need for efficient, scalable algorithms for *finite state automata* (FSA), by noting that they are usually the most computationally tractable form in which to analyze the regular languages that arise in many branches of computer science. That analysis requires efficient algorithms both for determinization of NFA (conversion of NFA to DFA) and minimization of DFA.

Recall that a *deterministic finite state automaton* (DFA) consists of a finite set of states with labelled, directed edges between pairs of states. The labels are drawn from an associated alphabet. For each state, there is at most one outgoing edge labelled by a given letter from the alphabet. So, a transition from a state dictated by a given letter is *deterministic*. There is an initial state and also certain of the states are called *accepting*. The DFA accepts a word if the letters of the word determine transitions from the initial state to an accepting state. The set of words accepted by a DFA is called a *language*.

A *non-deterministic finite state automaton* (NFA) is similar, except that there may be more than one outgoing edge with the same label for a given state. Hence, the transition dictated by the specified label is non-deterministic. The NFA accepts a word if there exists a choice of transitions from the initial state to some accepting state.

More formally, a DFA is a *5-tuple* $(\Sigma, Q, q_0, \delta, F)$, where Σ is the input alphabet, Q is the set of states of the automaton, $q_0 \in Q$ is the initial state, and there is a subset of Q, called the *final* or *accepting* states, F. $\delta : Q \times \Sigma \to Q$ is the *transition function*, which decides which state the control will move to from the current state upon consuming a symbol in the input alphabet.

An NFA is a *5-tuple* $(\Sigma, Q, q_0, \delta, F)$. The only difference from a DFA is that $\delta : Q \times \Sigma \to \mathfrak{P}(Q)$. Upon consuming a symbol from the input alphabet, an NFA can non-deterministically move control to any one of the defined next states.

Recall that the *subset construction* allows one to transform an NFA into a corresponding DFA that accepts the same words. Each state of the DFA is identified with a subset of the NFA states. Given a state A of the

DFA and an edge with label α, the destination state B consists of a subset of all states of the NFA having an incoming edge labelled by α and a source state that is a member of the subset A.

Finite state automata are an important computationally tractable representation of *regular languages*. This class of languages has a range of valuable closure properties, including under concatenation, union, intersection, complementation, reversal and the operations of (not necessarily deterministic) transducers. (A *transducer* is a DFA or NFA that also produces output letters upon each transition.) The above properties have algorithmic analogues that operate on finite state automata. For instance, given an FSA representing a language it is easy to construct one for the reversed language. So, one can compute various operations on regular languages by computing the analogous operations on their finite state automaton representations.

Using these operations to manipulate regular languages forces one to choose between a DFA and an NFA representation. But neither representation suffices. Some of the above operations on finite state automata, such as complementation, require input in the form of a DFA. And yet, some operations may transform a DFA into an NFA.

From a computability standpoint, there is no problem. The subset construction converts between an NFA and the more specialized DFA. But while the subset construction is among the best known algorithms of an undergraduate curriculum, it may also lead to an exponential growth in the number of states. This is usually the limiting factor in determining what computations are practical.

In some cases this problem is completely unavoidable, since there are families of non-deterministic automata whose languages cannot be recognized by any deterministic automata without exponentially many states. In many cases of interest, however, much smaller equivalent deterministic automata do exist. But the determinization process alone is not enough to reduce the DFA to the equivalent unique minimal DFA. No method is known of finding this minimal DFA without first constructing and storing the large intermediate DFA. It is this large intermediate data which motivates us to consider parallel disk-based computing.

Hopcroft [13] provided an efficient $O(n \log n)$ algorithm for DFA minimization, but the algorithm does not adapt well to parallel computing. An efficient parallel $O(n \log^2 n)$ algorithm has been used in the 1990s [31], but ultimately the lack of intermediate storage for the subset construction has prevented researchers from adapting these techniques for use within the varied applications described above.

4 Algorithms for NFA to Minimal DFA

This section presents disk-based parallel algorithms for both determinization (Section 4.1) and DFA minimization (Section 4.2), both using *streaming* access to data distributed evenly across the parallel disks of a cluster. This avoids the latency penalty that a random access to disk incurs. Separately, Section 4.3 presents a depth-first based algorithm for determinization and minimization suitable for large shared-memory computers.

For the parallel disk-based implementation, Roomy [20, 21] was used. Roomy is an open-source library for parallel disk-based computing, providing an API for operations with large data structures. Projects involving very large data structures have previously been successfully developed using various versions of Roomy: a parallel disk-based binary decision diagram package [23], or a parallel disk-based computation, which was used in 2007 to prove that any configuration of Rubik's cube can be solved in 26 moves or less [22]. The three Roomy data structures we used are the Roomy hash table, the Roomy list and the Roomy array. Each is a distributed data structure, which Roomy keeps load-balanced on the parallel disks of a cluster. Operations to these data structures are batched and delayed until the user decides that there are enough operations for processing to be performed efficiently. In doing so, a latency penalty is paid only once for accessing a large chunk of data, and aggregate disk bandwidth is significantly increased.

4.1 Subset construction for large NFAs

For subset construction on parallel disks, three Roomy-hash tables are used: $visited$, $frontier$ and $next\_frontier$. Hash table keys are sets of states of the NFA, and hash table values are unique integers. A hash table entry

is denoted as ($key \rightarrow value$). A Roomy-list of *triples* (set_{id}, $transition$, $next\_set_{id}$) is also used, to keep the already discovered part of the equivalent DFA.

The portion of any Roomy data structure d kept by a specific compute node k is denoted as d^k. Any Roomy data structure is load-balanced across the disks, so d^k and d^j will be of about the same size, for any compute nodes k, j.

Data that needs to be sent to other compute nodes by Roomy is first buffered in local RAM, in *buckets* corresponding to each compute node. For a given piece of data, Roomy uses a hash function to determine which compute node should process that data and, hence, in which bucket to buffer that data. Once a given buffer is full, the data it contains is sent over the network to the corresponding compute node (or to the local disk, if the data is to be processed by the local node).

Algorithm 1 Parallel Disk-based Subset Construction

Input: Initial NFA $init_{NFA}$, with initial state s_i and accepting states F_s, will be loaded in RAM on each of the N compute nodes.

Output: DFA $interm_{DFA}$, equivalent to $init_{NFA}$

Insert ($s_i \rightarrow new\ Id()$) in $visited$ and $frontier$. $next\_frontier$ is empty.
Each compute node k of the cluster does:
while $frontier^k$ is not empty **do**
  // Compute Neighbors of Frontier
  for each ($set \rightarrow set_{id}$) $\in frontier^k$ **do**
    for each transition T of $init_{NFA}$ **do**
      Apply T to each NFA state in set, to generate $next\_set$.
      $next\_set_{id} \leftarrow new\ Id()$
      Calculate $node$, the compute node responsible for the new NFA state, using a hash function ($1 \leq node \leq N$).
      Insert ($next\_set \rightarrow next\_set_{id}$) in a local RAM-based buffer $sets_{node}$.
      Insert triple ($set_{id}, T, next\_set_{id}$) in a local RAM-based buffer $triples_{node}$.
  // Scatter-Gather, when buffers are full
  for $k \in \{1 \ldots N\}$ **do**
    Send $sets_k$ and $triples_k$ to compute node k.
  for $k \in \{1 \ldots N\}$ **do**
    Receive a bucket of triples and a buckets of sets from each compute node k.
  // Duplicate Detection
  Aggregate received *sets* buffers in $next\_frontier^k$.
  Remove duplicate and previously visited sets from $next\_frontier^k$.
  Update all triples that correspond to a duplicate set.
  Add $next\_frontier^k$ to $visited^k$ and add all *triples* buffers to $triples^k$.
  $frontier^k \leftarrow next\_frontier^k$
Roomy-list *triples* now holds $interm_{DFA}$.
Convert Roomy-list *triples* into a compact Roomy-array-based DFA representation.

Parallel disk-based subset construction is described in Algorithm 1. Parallel breadth-first search (BFS) is used to compute the states of the intermediate DFA. Duplicate states in each BFS frontier are removed by delayed duplicate detection. The parallel disk-based computation follows a *scatter-gather* pattern in a loop: local batch computation of neighbors; send results of local computation to other nodes; receive results from other nodes; and perform duplicate detection. All parallel disk-based algorithms presented here (Algorithms 1, 3 and 4) use this kind of scatter-gather pattern.

4.2 Finding the Unique Minimal DFA

The algorithm used for computing the minimal DFA on parallel disks is based on a parallel RAM-based algorithm used on supercomputers in the late 1990s and early 2000s [16, 31, 37]. We call this the *forward refinement* algorithm. The central idea of the algorithm is to iteratively partition the states (to refine partitions of the states) of the given DFA, which is proven to converge to a stable set of partitions. Upon convergence, the set of partitions, together with the transitions between partitions, form a graph which is isomorphic to the minimal DFA. Initially, the DFA states are split into two partitions: the accepting states and the non-accepting states. A hash table of visited partitions, *parts*, is used, with pairs of integers as keys and integers as values. For the pair of integers, the first integer represents the partition number of the current state i, while the second integer represents the partition number of $DFA[i][T]$, where T is the current transition being processed. If two states i and j in the DFA are equivalent, then for any transition T, at any time during the iterative process, the pairs corresponding to i and j for the same T should have the same first integers and the same second integers. Algorithm 2 describes a sequential RAM-based version of *forward refinement*, while Algorithm 3 describes the parallel disk-based one.

Algorithm 2 Sequential RAM-based Forward Refinement

Input: A DFA $init_{DFA}$, with N states, with initial states I_s and accepting states F_s
Output: The minimal canonical DFA min_{DFA}, equivalent to $init_{DFA}$.

1: Initialize array $curr\_refs$: $curr\_refs[i] \leftarrow 0$ if i is a non-accepting state of $init_{DFA}$, and $curr\_refs[i] \leftarrow 1$ if i is an accepting state.
2: Initialize array $next\_refs$ to all 0.
3: $prev\_num\_refs \leftarrow 0; curr\_num\_refs \leftarrow 2$
4: **while** $prev\_num\_refs < curr\_num\_refs$ **do**
5:   $prev\_num\_refs \leftarrow curr\_num\_refs$
6:   **for** each transition T of $init_{DFA}$ **do**
7:     Initialize hash table $parts$ to $\emptyset$
8:     $next\_id \leftarrow 0$
9:     **for** $i \in \{1 \ldots N\}$ **do**
10:       $next\_part \leftarrow curr\_refs[init_{DFA}[i][T]]$
11:       $pair \leftarrow new\ Pair(curr\_refs[i], next\_part)$
12:       $id \leftarrow parts.getVal(pair)$
13:       **if** id was not found in $parts$ **then**
14:         Insert $(pair \rightarrow next\_id)$ in $parts$
15:         $id \leftarrow next\_id$
16:         $next\_id \leftarrow next\_id + 1$
17:       $next\_refs[i] \leftarrow id$
18:     $curr\_refs \leftarrow next\_refs$
19:   $curr\_num\_refs \leftarrow next\_id$
// For each state i of $init_{DFA}$, $curr\_refs[i]$ defines what partition state i is in.
20: Collapse each partition to just one state to obtain the minimal DFA.

The major differences between Algorithms 2 and 3 are that lines 7–17 and line 20 of Algorithm 2 are parallelized and that Roomy's principles of parallel disk-based computing are used: all large data structures are split into equally-sized chunks which are kept on the parallel disks of a cluster and all access and update operations to the *curr_refs* and *prev_refs* arrays and to the *parts* hash table are delayed and batched for efficient streaming access to disk. Also, duplicate detection, which in the sequential RAM-based algorithm appears in lines 12–17, is replaced by delayed duplicate detection.

Note that in Algorithm 3 each compute node k keeps its own part of the *parts* hash table ($parts^k$) and owns a part of the intermediate DFA states ($states^k$). As with subset construction, the parallel disk-based computation follows a scatter-gather pattern, denoted in the pseudocode by most of the *for* loops: local computation (lines 4–7), *scatter* (lines 8–9), *gather* (lines 10–11), local computation and *scatter* (lines

Algorithm 3 Parallel Disk-based Forward Refinement

Input: A DFA $init_{DFA}$, with N states, with initial states I_s and accepting states F_s
Output: The minimal canonical DFA min_{DFA}, equivalent to $init_{DFA}$.

1: // Initialization and outer loop are the same as lines 1–6 in Algorithm 2
2: // Disk-based parallel loop (parallelization of lines 7–17 in Algorithm 2) – each node k does:
3: Initialize hash table $parts^k$ to $\emptyset$
4: **for** $i \in \{states^k\}$ **do**
5:     $next\_part[i] \leftarrow curr\_refs[init_{DFA}[i][T]]$
6:     $pair[i] \leftarrow new\ Pair(curr\_refs[i], next\_part[i])$
7:     $next\_id[i] \leftarrow new\ Id()$
8: **for** $i \in \{states^k\}$ **do**
9:     Send new entry ($pair[i] \rightarrow next\_id[i]$) and state id i to node = $hash(pair[i])$
10: **for** $k \in \{1 \dots N\}$ **do**
11:     Receive $pair \rightarrow id$ entries from node k
12: **for** each received entry $pair \rightarrow recv\_id$ and associated state id i from a node k **do**
13:     **if** an entry $pair \rightarrow id$ was not found in the local $parts$ **then**
14:         Insert ($pair \rightarrow recv\_id$) in the local $parts$
15:         Send key–value pair $i \rightarrow recv\_id$ to node k
16:     **else**
17:         Send key–value pair $i \rightarrow id$ to node k
18: **for** $k \in \{1 \dots N\}$ **do**
19:     Receive $i \rightarrow id$ entries from node k
20: **for** each received entry $i \rightarrow id$ **do**
21:     $curr\_refs[i] \leftarrow id$

12–17), *gather* (lines 18–19) and local computation (lines 20–21).

The last part of finding the minimal DFA, in which each partition collapses to one state, is presented separately, in Algorithm 4.

Algorithm 4 Parallel Disk-based Partitions Collapse

// Collapsing partitions to min_{DFA} (parallelization of line 20 in Algorithm 2) – each node k does:
for $i \in \{indices^k\}$ **do**
    Get $partition[i]$ (the partition of state i) from $curr\_refs^k$
    for each transition T of $init_{DFA}$ **do**
        Get $partition[init_{DFA}[i][T]]$ from node that owns it
    // Now all the transitions of $partition[i]$ in min_{DFA} are known

4.3 Multi-threaded Implementations for Shared Memory

For comparison with the parallel disk-based algorithms, multi-threaded shared-memory implementations of subset construction and DFA minimization are provided. A shared-memory architecture almost always has less storage (128 GB RAM in our experiments) than parallel disks. To alleviate the state combinatorial explosion issue, depth-first search (DFS) is used here for the subset construction instead of breadth-first search (BFS).

The smallest instance of the four NFA to minimal DFA problems considered can be solved on a commodity computer with 16 GB of RAM, the second instance needs 40 GB of memory for subset construction, while the third largest instance needs a large shared-memory machine with at least 100 GB of RAM. The largest instance considered cannot be solved even on a large shared-memory machine, thus requiring the use of parallel disks on a cluster.

A significant problem for both subset construction and DFA minimization in a multi-threaded environment is synchronization for duplicate detection. For subset construction, this issue arises when a thread discovers a new DFA state and checks whether the state has been discovered before by itself or another thread. The data structure keeping the already-discovered states (usually a hash table) has to be locked in that case, so that the thread can check whether the current state has already been discovered and, if not, to insert the new state in the data structure. However, such an approach would lead to excessive lock contention (many threads waiting on the same lock).

Hence, the solution employed was to use a partitioned hash table to keep the already discovered states instead of a regular hash table. For large problem instances, the hash table was partitioned into 1024 separate hash tables — each with its own lock. So long as the number of partitions is much larger than the number of threads, it is unlikely that two threads will concurrently discover two states that will belong to the same partition, thus avoiding most of the lock contention. Experiments (see Section 6, Table 4) show significant speedup with the increase in number of threads.

A similar solution was used for the forward refinement algorithm, which minimizes the DFA obtained from subset construction. In this case, read accesses significantly dominate over write accesses. The implementation took advantage of this by implementing a lock only around writes to the corresponding hash table. The valid bit was written last in this case. A write barrier is needed to guarantee no re-ordering of writes. In the worst case, a concurrent read may read the hash entry as invalid, and that thread will then request the lock, verify that the hash entry is still invalid, and if that is the case, then do the write. This is safe.

5 Token Passing Networks

Section 5.1 provides background on token passing networks, and the specific challenge problem addressed here. Section 5.2 describes the computation on token passing networks addressed here and the component of that computation that requires the parallel solutions of this paper.

5.1 Stacks, Token Passing Networks and Forbidden Patterns

The study of what permutations of a stream of input objects could be achieved by passing them through various data structures goes back at least to Knuth [17, Section 2.2.1], who considered the case of a stack and obtained a simple characterization and enumeration in this case. Knuth's characterization uses the notion of *forbidden substructures*: a permutation can be achieved by a stack if and only if it does not contain any three numbers (not necessarily consecutive) whose order within the permutation, and relative values match the pattern high-low-middle (usually written 312). For instance 41532 cannot be achieved because of 4, 1 and 2. This work has spawned a significant research area in combinatorics, the study of permutation patterns [24] in which much beautiful structure has been revealed. Nevertheless, many problems very close to Knuth's original one remain unresolved: in particular there is no similar characterization or enumeration of the permutations achievable by two stacks in series (it is not even known if 2-stack achievability can be tested in polynomial time). A number of authors have investigated restricted forms of two-stack achievability [8] including the case of interest here, where the stacks are restricted to finite capacity, in which case they can be modelled as *token passing networks*, as introduced in [5].

To recap briefly, a token passing network is a directed graph with designated input and output vertices. Numbered tokens are considered to enter the graph one at a time at the input vertex, and travel along edges in the appropriate direction. At most one token is permitted at any vertex at any time. The tokens leave the graph one at a time at the output vertex. A permutation $\pi \in S_n$ is called *achievable* for a given network if it is possible for tokens to enter in the order $1, \ldots, n$ and leave in the order $1\pi, \ldots, n\pi$.

In this case, the two stacks can be modelled as a finite token passing network (as seen, for example in Figure 1) and their behavior studied using the techniques of [5]. These techniques allow the classes of achievable permutations and the forbidden patterns that describe them to be encoded by regular languages

and manipulated using finite state automata using a collection of GAP [11] programs developed by the fourth author and M. Albert.

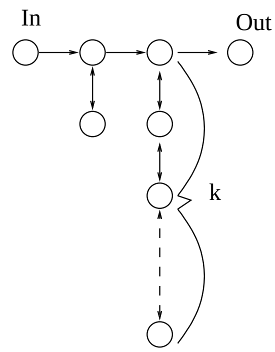

Figure 1: A 2-stack followed by a k-stack represented as a token passing network

In previous work, the fourth author explored the cases of stacks of depths 2 and depth k (as seen in Figure 1) for a range of values of k and observed that for large enough k the sets of minimal forbidden patterns appeared to converge to a set of just 20 of lengths between 5 and 9, which were later proved [10] to describe the case of a 2-stack and an infinite stack.

The application that motivates the calculations in this paper is a step towards extending this result to a 3-stack and an infinite stack, by way of the slightly simpler case of a 3-buffer (a data structure which can hold up to three items and output any of them). This configuration is shown in Figure 2.

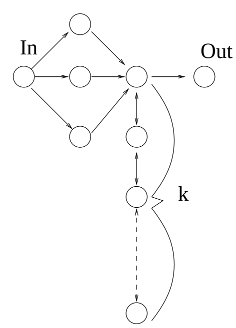

Figure 2: A 3-buffer followed by a k-stack represented as a token passing network

Computations had been completed on various sequential computers for a 3-buffer and a k-stack for $k \leq 8$, but this was not sufficient to observe convergence. The examples considered in this paper are critical steps in the computations for $k = 9$, $k = 10$, $k = 11$ and $k = 12$. Based on the results of these computations we are now able to conjecture with some confidence a minimal set of 12,636 forbidden permutations for a 3-buffer and an infinite stack of lengths between 7 and 18.

5.2 The Computation

The computations required for these investigations are those implied by Corollary 1 of [2, p. 96]. By modelling the token passing network in the style of [5], slightly optimized to avoid constructing so many redundant states, we can construct (an automaton representing) a language L describing the permutations achievable by our network, and we wish to construct a language B describing the minimal forbidden patterns. Each

state of L represents a configuration of the network and the labels on the transitions represent (rank encoded) output symbols, if any. Combining results from [2] and simplifying the notation a little we find

$$B = \left(L^{RC} \cap \left(L^{RC} D^t\right)^C\right)^R$$

where R denotes left-to-right reversal, C denotes complementation and D is the deletion transducer described in [2]. Each step of this computation can be realized by standard algorithms using finite state automata, but, as observed above, with frequent recourse to determinization (to allow complements) and minimization (to control explosion in the number of states). As the computations become larger, the limiting step turns out to arrive after the application of the transposed deletion transducer and before the next complementation, and it is this step that we have parallelized in this paper.

6 Experimental Results

6.1 Parallel Disk-based Computations

Parallel disk-based computations were carried out on a 29-node computer cluster, each node's processor being a 4-core Intel Xeon CPU 5130 running at 2 GHz. Nodes on the cluster had either 8 or 16 GB of RAM and at least 200 GB of free disk storage and ran Red Hat Linux kernel version 2.6.9.

Table 1 presents the sizes of the intermediate DFAs produced by subset construction and the sizes of the minimal DFAs produced by the minimization process for the four considered token passing network problems (corresponding to stack depths 9, 10, 11 and 12).

Table 1: Solutions for the four considered problems.

Stack depth	**NFA size** (#states)	**Interm. DFA size** (#states)	**Min. DFA size** (#states)
9	167,143	49,722,541	32,561
10	537,294	175,215,168	95,647
11	1,667,428	587,547,014	274,752
12	5,035,742	1,899,715,733	774,172

Table 2 shows the running time and aggregate disk-space used by the subset construction results for the four problem instances. Each state in the intermediate DFA is a subset of states in the original NFA and needs to be kept as such until the subset construction phase is over, for the purpose of exact duplicate detection. Hence, for each newly discovered DFA state, the entire corresponding subset needs to be stored on disk. The average subset size (the sum of all subset sizes divided by the number of subsets) increases slightly with stack depth, from an average of 8.48 states per set for stack depth 8 to 10.06 states per set for stack depth 12.

Table 2: Parallel disk-based subset construction.

Stack depth	**NFA size** (#states)	**Intermediate DFA** **Size** (#states)	**Peak disk**	**Time**
9	167,143	49,722,541	24 GB	9min
10	537,294	175,215,168	90 GB	29min
11	1,667,428	587,547,014	327 GB	3h 40min
12	5,035,742	1,899,715,733	1,136 GB	1day 12h

Figure 3 presents the breadth-first search frontier sizes for the largest case ($k = 12$). This and the other three cases exhibit a thin bell-shaped curve, in contrast to the pear-shaped curve seen for many other implicit graph enumerations.

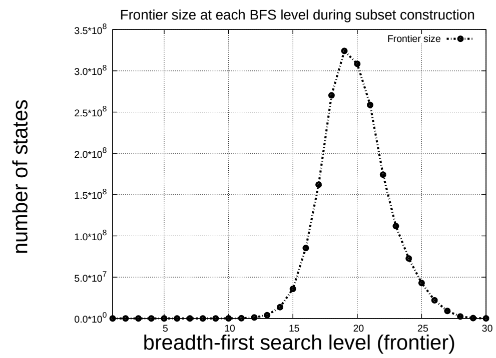

Figure 3: Frontier sizes for each BFS level of the implicit graph corresponding to subset construction.

The intermediate DFA (produced by subset construction) was then minimized using the forward refinement algorithm. Experimental results for DFA minimization are presented in Table 3. For each of the four problem instances, the computation required exactly five forward refinements (five of the outer iterations described in Algorithm 3).

Table 3: Parallel disk-based DFA minimization results.

Stack depth	**Num. trans.**	**Interm. DFA** **Size** (#states)	**Minimal DFA** **Peak disk**	**Time**
9	11	49,722,541	6 GB	38min
10	12	175,215,168	22 GB	2h 42min
11	13	587,547,014	81 GB	9h 20min
12	14	1,899,715,733	295 GB	1day 8h

The DFA minimization times, reported in Table 3, grow steadily, almost linearly, with the increase in number of states of the intermediate DFA. On the other hand, the subset construction times from Table 2 increase much more rapidly. There are two reasons for this. First, the two smaller cases run faster because the distributed subset construction fits in the aggregate RAM of the nodes of the cluster. Second, we suspect the computation to be network-limited. The cluster is five years old and uses the 100 Mb/s (12.5 MB/s) Fast Ethernet commodity network of that time. This point-to-point network speed is significantly slower than disk. This especially penalizes the two larger cases.

6.2 Multi-threaded RAM-based Computations

Multi-threaded computations were run on a large shared-memory machine with four quad-core 1.8 GHz AMD Opteron processors (16 cores), 128 GB of RAM, running Ubuntu 9.10 with a SMP Linux 2.6.31 server kernel.

Only the first three computations could be completed on the large shared-memory machine used. The fourth computation requires far too much memory. Table 4 shows how the running time of the subset construction and DFA minimization scales with the number of worker threads. The reported timings are for the stack depth 11 problem instance. The size of the intermediate DFA produced by subset construction for this instance is 587,547,014 states. The minimal DFA produced by forward refinement has 274,752 states. For any number of worker threads, the peak memory usage for subset construction was 98 GB, while for minimization it was 36.5 GB.

The timings in Table 4 show that both the multi-threaded subset construction and the DFA minimization implementations scale almost linearly with the number of threads. DFA minimization scales almost linearly

Table 4: Multi-threaded RAM-based timings for stack depth 11.

	Num. threads				
Subset	1	2	4	8	16
constr.	15h 30min	8h 10min	3h 50 min	2h 5min	1h 15min
Minimiz.	8h	5h 5min	2h 40 min	1h 25min	57min
Total	23h 30min	13h 15min	6h 30 min	3h 30min	2h 12min

for up to 8 threads. From 8 to 16 threads it scales sub-linearly due to significant lock contention.

Table 5 presents the timings for the two smallest instances when using 16 worker threads. For the stack depth 9 case, the peak memory usage was 12 GB for subset construction and 5 GB for DFA minimization. For stack depth 10, the peak memory usage was 40 GB and 11 GB, respectively.

Table 5: Multi-threaded RAM-based results for stack depths 9 & 10, with 16 worker threads.

Stack depth	**Time**		
	Subset constr.	**DFA min.**	**Total**
9	8 min	4min 10s	12min 10s
10	25 min	15min	40min
11	1 hr 15 min	57min	2hr 12min

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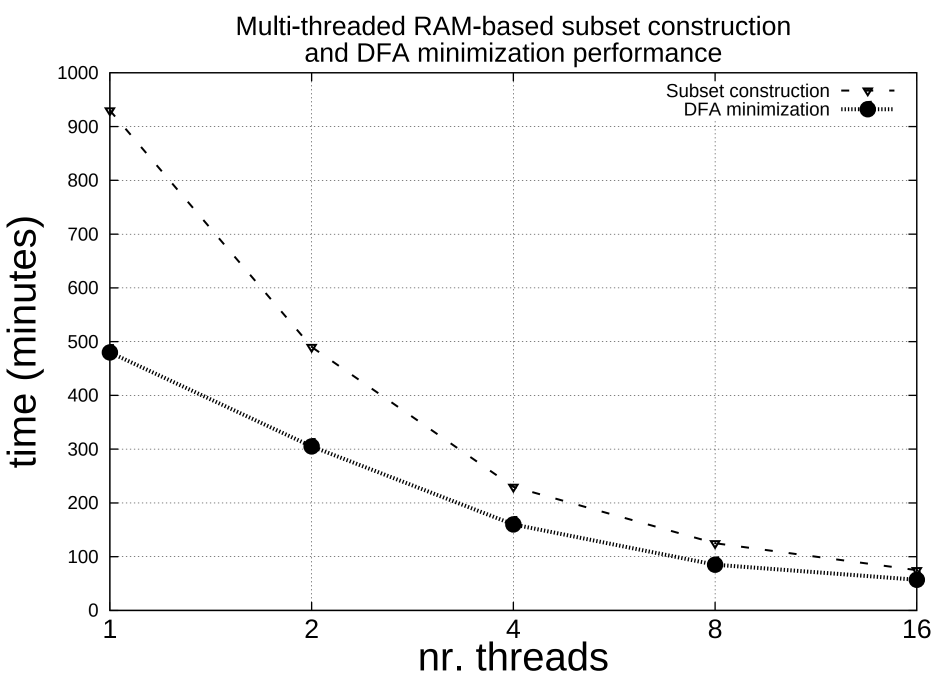

Multi-threaded RAM-based subset construction
and DFA minimization performance
Subset construction
DFA minimization
time (minutes)
nr. threads
0
100
200
300
400
500
600
700
800
900
1000
1
2
4
8
16